Holographic Confinement/Deconfinement Phase Transitions of AdS/QCD in Curved Spaces

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Abstract

Recently Herzog has shown that deconfinement of AdS/QCD can be realized, in the hard-wall model where the small radius region is removed in the asymptotically AdS space, via a first order Hawking-Page phase transition between a low temperature phase given by a pure AdS geometry and a high temperature phase given by the AdS black hole in Poincare coordinates. In this paper we first extend Herzog's work to the hard wall AdS/QCD model in curved spaces by studying the thermodynamics of AdS black holes with spherical or negative constant curvature horizon, dual to a non-supersymmetric Yang-Mills theory on a sphere or hyperboloid respectively. For the spherical horizon case, we find that the temperature of the phase transition increases by introducing an infrared cutoff, compared to the case without the cutoff; For the hyperbolic horizon case, there is a gap for the infrared cutoff, below which the Hawking-Page phase transition does not occur. We also discuss charged AdS black holes in the grand canonical ensemble, corresponding to a Yang-Mills theory at finite chemical potential, and find that there is always a gap for the infrared cutoff due to the existence of a minimal horizon for the charged AdS black holes with any horizon topology.

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1 Introduction

The remarkable AdS/CFT correspondence [1] conjectures that string/M theory in an anti-de Sitter space (AdS) times a compact manifold is dual to a large N strongly coupling conformal field theory (CFT) residing on the boundary of the AdS space. A special example of the AdS/CFT correspondence is that type IIB string theory in $AdS_5 \times S^5$ is dual to a four dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory on the boundary of AdS_5. At finite temperature, in the spirit of the AdS/CFT correspondence, Witten [2] has argued that the thermodynamics of black holes in AdS space can be identified with that of the dual strongly coupling field theory in the high temperature limit. Therefore one can discuss the thermodynamics and phase structure of strongly coupling field theories by studying the thermodynamics of various kinds of black holes in AdS space. Indeed, it is well-known that there exists a phase transition between the Schwarzschild-AdS black hole and thermal AdS space, the so-called Hawking-Page phase transition [3]: the black hole phase dominates the partition function in the high temperature limit, while the thermal AdS space dominates in the low temperature limit. This phase transition is of first order, and is interpreted as the confinement/deconfinement phase transition in the dual field theory [2]. Note that the dual conformal field theory to the Schwarzschild-AdS black hole configuration resides on a sphere. For example, for the five dimensional Schwarzschild-AdS black holes, the dual field theory is $\mathcal{N}=4$ supersymmetric Yang-Mills theory at finite temperature [2]. That is, by discussing the thermodynamics of AdS black holes with a spherical horizon, one can study thermodynamics of the dual non-supersymmetric Yang-Mills theory on a sphere at finite temperature.

It is interesting to note that in AdS space the black hole horizon is not necessarily a sphere [4], the black hole horizon can also be a Ricci flat surface [5] or a negative constant curvature surface [6]. Thus using those AdS black hole solutions, one can study dual strongly coupled field theories residing on Ricci flat or hyperbolic spaces. These so-called topological black holes have been investigated in higher dimensions [7, 8, 9, 10] and in dilaton gravity [11, 12]. It was found that the Hawking-Page phase transition, which happens for spherical AdS black holes, does not occur for Ricci flat and negative curvature AdS black holes, the latter two being not only locally stable (heat capacity is always positive), but also globally stable (see for example, [7]). Here it is worth mentioning that if some directions of the horizon surface are compact for Ricci flat AdS black holes, similar to the case of spherical AdS black holes, the Hawking-Page phase transition can occur [13, 14, 15, 16, 17] due to the existence of so-called AdS soliton [18]. The absence of a Hawking-Page phase transition for Ricci flat AdS black holes without compact directions

is consistent with the result that the AdS space and the AdS black holes in Poincare coordinates without compact directions are both in the deconfinement phases [2]. This can be confirmed by calculating quark-anti-quark potential through Wilson-loop [19]. In addition, the black hole entropy is proportional to N^2, where N is the rank of gauge group $SU(N)$ for the dual gauge field.

Using the AdS/CFT correspondence, it is expected that we can obtain some more qualitative understanding of QCD and the nature of confinement. The authors of papers [20, 21, 22] are able to realize confinement of related supersymmetric field theories by finding dual gravitational configurations where the geometry in the infrared at small radius is capped off in a smooth way. The author of [23] proposed a simpler model, AdS_5 in Poincare coordinates without compact directions where the small radius region is removed, to realize confinement. The field theory dual to this is a non-supersymmetric Yang-Mills theory in four dimensions. Although such a model is somewhat rough, the results obtained in [24, 25] show that one can get some realistic, semiquantitative descriptions of low energy QCD, by using this hard wall model.

Recently, Herzog [26] (see also [27]) has shown that in this simple hard wall model of AdS/QCD, the confinement/deconfinement phase transition occurs via the first order Hawking-Page phase transition between the low temperature thermal AdS space and high temperature AdS black hole in Poincare coordinates. Note the facts that removing the small radius region of AdS_5 is dual to introducing an infrared (IR) cutoff in the dual field theory and that the Hawking-Page phase transition does not occur for the negative constant curvature AdS black holes without a cutoff. It is quite interesting to revisit the thermodynamics and the Hawking-Page phase transitions for spherical AdS black holes and negative constant curvature AdS black holes by removing the small radius regions. That is, we are interested in the confinement/deconfinement phase transition for the hard wall QCD model in curved spaces. Recall that charged AdS black hole configurations are dual to supersymmetric field theories with so-called R-charges [28, 29, 30]. Introducing R-charges to the AdS black holes is equivalent to introducing a chemical potential in the dual field theory. In this paper, therefore, we will also discuss the case of charged AdS black holes.

The paper is organized as follows. In the next section we will revisit the Hawking-Page phase transition for spherical AdS black holes with an IR cutoff. The case for the negative constant curvature black holes will be discussed in Sec. 3. In. Sec. 4 we will investigate the phase transition for charged AdS black holes with an IR cutoff. Sec. 5 will be devoted to conclusions and discussions.

2 Hawking-Page Phase Transition for Spherical Black Holes with an IR Cutoff

Let us start with the action of five-dimensional general relativity with a negative cosmological constant

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left(R + \frac{12}{l^2}\right), \tag{2.1}$$

where $k^2 = 8\pi G$ and l is the radius of the five-dimensional AdS space. The action (2.1) admits the AdS spherical black hole solution with the metric

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_3^2, \tag{2.2}$$

where

$$V(r) = 1 + \frac{r^2}{l^2} - \frac{r_s^2}{r^2}, \tag{2.3}$$

$d\Omega_3^2$ is the line element of a three-dimensional sphere with unit radius and r_s^2 is an integration constant, the Schwarzschild mass parameter. The black hole horizon is determined by $V(r_+) = 0$, which gives $r_+^2 = l^2(-1 + \sqrt{1 + 4r_s^2/l^2})/2$. The black hole has an inverse temperature $1/T$

$$\beta = 1/T = \frac{2\pi r_+}{1 + 2r_+^2/l^2}. \tag{2.4}$$

In the global coordinate (2.2), the AdS boundary is located at $r \to \infty$. In Poincare coordinates the radial direction is dual to the energy scale of the dual field theory. If one introduces a regulating UV cutoff at $r = R \gg r_+$, then the range from $r = R$ to ∞ should be removed from the bulk geometry. On the other hand, one further introduces an IR cutoff at $r = r_0$ in the dual field theory (The IR cutoff is equivalent to a mass gap in the dual field theory), the range from $r = 0$ to r_0 also should be removed from the bulk geometry [23]. This corresponds to a hard wall truncation of the space, independent of the radius of the horizon of the black hole. Namely, we are considering the bulk geometry from $r = r_0$ to $r = R$. The UV cutoff is necessary in order to regularize the action. At the end of calculation, we will remove the UV cutoff by letting $R \to \infty$.

To see the phase structure of QCD in the hard wall AdS/QCD model, let us calculate the Euclidean action of the AdS black hole, by choosing the AdS vacuum solution (2.2) with

$$V_b = 1 + \frac{r^2}{l^2} \tag{2.5}$$

as the reference background. As a result, the Euclidean action of the reference background is

$$\mathcal{S}_b = \frac{4\Omega_3}{\kappa^2 l^2}\beta_b\int_{r_0}^{R} r^3 dr, \tag{2.6}$$

while the Euclidean action of the black hole is

$$\mathcal{S}_{bh} = \frac{4\Omega_3}{\kappa^2 l^2}\beta \int_{r_{\max}}^{R} r^3 dr, \tag{2.7}$$

where Ω_3 is the volume of the unit three-sphere and $r_{\max} = \max(r_0, r_+)$ is the IR cutoff in the high temperature phase. If $r_0 < r_+$, one has $r_{\max} = r_+$ while $r_{\max} = r_0$ as $r_0 > r_+$. In addition, let us mention here that usually the Hilbert-Einstein action of general relativity should be supplemented by the Gibbons-Hawking surface term in order to have a well-defined variational principle. However, for the case of the asymptotically AdS spaces, such terms have no contribution to the difference of two Euclidean actions between the configuration under consideration and the reference background [2].

In order that the black hole solution (2.2) can be embedded into the background consistently, at the boundary $r = R$ the period β_b of the Euclidean time for the reference background has to obey the relation

$$\beta_b \sqrt{V_b(R)} = \beta \sqrt{V(R)}. \tag{2.8}$$

Using (2.8), calculating the difference between (2.7) and (2.6), and taking the limit $R \to \infty$, we obtain

$$\mathcal{I} = -\frac{\Omega_3 \beta}{2\kappa^2 l^2}\left(2r_{\max}^4 - r_+^4 - 2r_0^4 - r_+^2 l^2\right). \tag{2.9}$$

Without the IR cutoff, namely $r_0 = 0$, the action (2.9) reduces to

$$\mathcal{I} = -\frac{\Omega_3 \beta}{2\kappa^2 l^2} r_+^2 (r_+^2 - l^2). \tag{2.10}$$

Clearly when $r_+ = l$, the Euclidean action alters its sign, a first-order phase transition happens. This is just the well-known Hawking-Page phase transition. The phase transition temperature is

$$T_{\rm HP} = \frac{3}{2\pi l}. \tag{2.11}$$

When $r_+ > l$, or $T > T_{\rm HP}$, the black hole phase is dominant in the partition function, while the thermal AdS space is dominant as $T < T_c$ in the low temperature phase. With the action (2.10), we obtain the mass of the AdS black hole, via $E = \partial\mathcal{I}/\partial\beta$,

$$E = \frac{3\Omega_3 r_+^2}{2\kappa^2}\left(1 + \frac{r_+^2}{l^2}\right) = M, \tag{2.12}$$

and the entropy of the black hole, via $S = \beta E - \mathcal{I}$,

$$S = \frac{\Omega_3 r_+^3}{4G}, \tag{2.13}$$

satisfying the area formula of black hole entropy.

Now we consider the case with an IR cutoff, r_0. There are two quite different cases.

(i) $r_0 \geq r_+$. In this case, one has $r_{\rm max} = r_0$. And the action (2.9) turns out to be

$$\mathcal{I} = \frac{\Omega_3 \beta}{2\kappa^2 l^2} \left(r_+^4 + r_+^2 l^2 \right). \tag{2.14}$$

Clearly in this case, the action is always positive and therefore no phase transition will occur. The thermal AdS space is globally stable and is dominant in the partition function of the dual field theory.

(ii) $r_0 < r_+$. In this case one has $r_{\rm max} = r_+$. And the action (2.9) becomes

$$\mathcal{I} = -\frac{\Omega_3 \beta}{2\kappa^2 l^2} \left(r_+^4 - 2r_0^4 - r_+^2 l^2 \right). \tag{2.15}$$

Obviously, the action changes its sign at

$$r_+^2 = \frac{l^2}{2} \left(1 + \sqrt{1 + 8r_0^4/l^4} \right). \tag{2.16}$$

Therefore in this case, the phase transition can occur and the critical temperature is

$$T_c = \frac{2 + \sqrt{1 + 8r_0^4/l^4}}{\sqrt{2}\pi l \sqrt{1 + \sqrt{1 + 8r_0^4/l^4}}} > T_{\rm HP}. \tag{2.17}$$

This critical temperature is higher than the one for the case without the IR cutoff. The thermal energy of dual QCD is

$$E = M + \frac{\Omega_3 r_0^4}{\kappa^2 l^2}, \tag{2.18}$$

where M is given in (2.12), and the entropy is still given by (2.13). We can clearly see from (2.18) the relation between the IR cutoff r_0 and the mass gap in the holographic QCD..

As a result, we have seen that by introducing an IR cutoff, the critical temperature for the corresponding Hawking-Page phase transition increases, compared to the case without the cutoff. The critical temperature is determined by the ratio r_0/l. In addition, we stress here that the condition $r_0 < r_+$ in the gravity side corresponds to the one that the mass gap is less than temperature in the field theory side. Similar statement holds in the following discussions.

3 Hawking-Page Phase Transition for Hyperbolic Black Holes with an IR Cutoff

In this section we will consider the so-called negative curvature black holes whose horizon is a negative constant curvature surface. In this case, the AdS black hole has the form

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Sigma_3^2, \tag{3.1}$$

where

$$V(r) = -1 + \frac{r^2}{l^2} - \frac{r_s^2}{r^2}, \tag{3.2}$$

and $d\Sigma_3^2$ is the line element for the three-dimensional surface with curvature -6. With suitable identification, one can construct closed horizon surfaces with high genus. Such black holes are called topological black holes in some of the literature. In this case, the dual field theory resides on a hyperboloid described by $d\Sigma_3^2$. In this negative curvature case, there exist two remarkable features of the solution. One is that even when the mass parameter r_s^2 vanishes, the metric (3.1) still has a black hole causal structure. The metric function $V(r)$ is replaced by

$$V_b = -1 + \frac{r^2}{l^2}. \tag{3.3}$$

The black hole is called a "massless" black hole with a horizon at $r_+ = l$. The other is that when r_s^2 is negative, there exists a "negative mass" black hole with

$$V_b(r) = -1 + \frac{r^2}{l^2} + \frac{r_b^2}{r^2}, \tag{3.4}$$

provided $0 < r_b^2 \leq l^2/4$. When $r_b = l/2$, the black hole solution (3.4) is an extremal one with vanishing temperature. For the black hole (3.2), the inverse Hawking temperature is

$$\beta = \frac{2\pi r_+}{2r_+^2/l^2 - 1}, \tag{3.5}$$

where r_+ is the black hole horizon satisfying $V(r_+) = 0$. Once again, in order to regularize the Euclidean action, one has to choose a suitable reference background. One may consider the AdS space (3.3) as the reference background. But its shortcoming is that the period of the Euclidean time of the solution is fixed as $2\pi l$ and cannot be chosen as an arbitrary value, otherwise the reference background has a conical singularity. Another choice is the solution (3.4) with $r_b = l/2$. In this case, the Euclidean time can be arbitrary since it describes an extremal black hole. Of course, when $r_b = 0$, the solution (3.4) reduces to (3.3). Therefore in what follows, we will consider the solution (3.4) with $r_b = l/2$ as the

reference background. When $r_b = 0$, it becomes the case (3.3). That is, in the following discussions, either $r_b = l/2$ or $r_b = 0$.

Considering the period β_b of the Euclidean time for the reference background obeying

$$\beta_b\sqrt{V_b(R)} = \beta\sqrt{V(R)}, \tag{3.6}$$

at the boundary, and introducing an IR cutoff r_0 (here the meaning of r_0 is the same as the one in the previous section, the relation of r_0 to the mass gap in the dual field theory can be obtained by calculating the mass from the Euclidean action below), we obtain the Euclidean action difference between the black hole and the background

$$\mathcal{I} = -\frac{\Sigma_3\beta}{2\kappa^2 l^2}\left(2r_{\rm max}^4 - 2r_0^4 - r_+^4 + r_+^2 l^2 - r_b^2 l^2\right), \tag{3.7}$$

where Σ_3 is the volume of the closed horizon surface with unit radius. First let us consider the case without the cutoff. In that case, $r_{\rm max} = r_+$, and the action reduces to

$$\mathcal{I} = -\frac{\Sigma_3\beta}{2\kappa^2 l^2}\left(r_+^4 + r_+^2 l^2 - r_b^2 l^2\right). \tag{3.8}$$

Note that the minimal horizon radius for the black hole (3.2) is $r_{\rm min} = l/\sqrt{2}$. If one chooses $r_b = l/2$ and (3.4) as the reference background, or $r_b = 0$ and (3.3) as the reference background, then the action (3.8) is always negative and the dual field theory is in the deconfinement phase. Therefore the usual Hawking-Page phase transition will not occur in this case.

Next we consider the case with an IR cutoff r_0. Here there are also two different cases.

(i) $r_0 > r_+$. In this case, one has $r_{\rm max} = r_0$, and the action becomes

$$\mathcal{I} = \frac{\Sigma_3\beta}{2\kappa^2 l^2}\left(r_+^4 - r_+^2 l^2 + r_b^2 l^2\right). \tag{3.9}$$

If $r_b = l/2$, we find that the action is always positive. Thus no phase transition occurs. If $r_b = 0$, however, an interesting phenomenon appears. For those "negative mass" black holes with horizon radius $l/\sqrt{2} \leq r_+ < l$, the action is negative, while it is positive for black holes with horizon $r_+ > l$. The action changes its sign at $r_+ = l$. This implies that in the low temperature phase $0 < T < 1/2\pi l$, the system is globally stable, while it becomes unstable as $T > 1/2\pi l$; a Hawking-Page phase transition happens at $r_+ = l$. This is an anti-intuitive result. This seemingly indicates that choosing the AdS black hole solution (3.3) is not suitable. Indeed, the result from the surface counterterm method indicates that one should choose $r_b = l/2$ as the reference background [31].

(ii) $r_0 < r_+$. In this case, we have $r_{\rm max} = r_+$, and the action is given by

$$\mathcal{I} = -\frac{\Sigma_3\beta}{2\kappa^2 l^2}\left(r_+^4 - 2r_0^4 + r_+^2 l^2 - r_b^2 l^2\right). \tag{3.10}$$

When $r_+^2 > r_c^2$ with

$$r_c^2 = \frac{l^2}{2}\left(-1+\sqrt{1+\frac{8r_0^4}{l^4}+\frac{4r_b^2}{l^2}}\right), \tag{3.11}$$

the action is negative, while it is positive for $r_{\min}^2 \le r_+^2 < r_c^2$. The Hawking-Page phase transition happens at $r_+ = r_c$. The critical temperature is

$$T_c = \frac{2r_c^2/l^2 - 1}{2\pi r_c}. \tag{3.12}$$

Note that when $r_0 = 0$, one has $r_c < r_{\min}$, the action is always negative. In order to have $r_c > r_{\min}$, there exists therefore a gap for the IR cutoff r_0:

$$\frac{r_0^4}{l^4} > \frac{1}{8}\left(3 - \frac{4r_b^2}{l^2}\right) = \frac{1}{4}. \tag{3.13}$$

In the above calculation we have taken $r_b = l/2$. Therefore the IR cutoff must be larger than the minimal horizon radius $r_{\min} = l/\sqrt{2}$.

On the other hand, if we take $r_b = 0$, the critical radius is

$$r_c^2 = \frac{l^2}{2}\left(-1+\sqrt{1+\frac{8r_0^4}{l^4}}\right). \tag{3.14}$$

The Hawking-Page phase transition happens when the temperature crosses the critical value (3.12). In this case, there is no gap for the IR cutoff r_0.

4 Hawking-Page Phase Transition for Charged AdS Black Holes with an IR Cutoff

In this section we consider the cases where the dual gravity configurations are charged AdS black holes with different topology horizons. In this case, the dual field theory is a Yang-Mills theory with chemical potential at finite temperature. We start with the Einstein-Maxwell action with a negative cosmological constant in five dimensions

$$\mathcal{S} = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left(R + \frac{12}{l^2} - F^2\right), \tag{4.1}$$

where $F_{\mu\nu}$ is the Maxwell field strength. This action can come from the spherical (S^5) reduction of type IIB supergravity [28, 29, 30]. The charged AdS black holes have the metric form

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_3^2, \tag{4.2}$$

where

$$V(r) = k - \frac{m}{r^2} + \frac{q^2}{r^4} + \frac{r^2}{l^2}, \tag{4.3}$$

where $d\Omega_3^2$ is the line element for a three-dimensional surface with constant curvature $6k$. Without loss of generality, one may take $k = 1$, 0 or -1. In addition, m and q are two integration constants, which are related to the mass and charge of the solution, respectively.

The mass parameter m can be expressed in terms of the horizon, r_+, as $m = r_+^2(k + q^2/r_+^4 + r_+^2/l^2)$. In order that the metric (4.3) describes a black hole with horizon r_+, the potential must have a positive radial derivative leading to the constraint on the horizon radius

$$2r_+^6 + kr_+^4 l^2 - q^2 l^2 \geq 0, \tag{4.4}$$

otherwise, the solution describes a naked singularity. This condition can also be obtained from following Hawking temperature of the black hole (to keep the positiveness of the Hawking temperature). The Hawking temperature of the black hole is

$$T = \frac{1}{\beta} = \frac{1}{2\pi r_+}\left(k + \frac{2r_+^2}{l^2} - \frac{q^2}{r_+^4}\right). \tag{4.5}$$

For the charged solution, the associated electric potential is

$$\mathcal{A} = \left(-\frac{q}{cr^2} + \Phi\right) dt, \tag{4.6}$$

where $c = 2/\sqrt{3}$, and Φ is a constant. We choose a gauge with $\Phi = q/cr_+^2$ so that the potential vanishes at the black hole horizon. Note that this provides us with a gauge invariant quantity because Φ describes a potential difference between at the horizon and at the infinity. Namely, if one chooses another gauge, Φ appearing in the following equations represents the potential difference between at the horizon and at the infinity. As a result, Φ is gauge invariant.

To discuss the thermodynamics and phase structure of dual QCD, one has to choose a suitable reference background. For the charged solution (4.3), one suitable background is the solution (4.2) with

$$V_b(r) = k + \frac{r^2}{l^2} + \frac{r_b^2}{r^2}\delta_{k,-1}, \tag{4.7}$$

where $r_b = l/2$ or 0, based on the discussion in the previous section. The choice of the background implies that we are going to discuss the thermodynamics of the charged black holes in a grand canonical ensemble, where the temperature and electric (chemical) potential are fixed at the boundary. Note that although the solution (4.7) is the one without

charge, we are discussing the thermodynamics in grand canonical ensemble, thus, the solution (4.7) with a constant potential Φ is still a solution of the action (4.1). Therefore the solution (4.7) with a constant potential is a good reference background, in order to analyze the thermodynamics of charged black holes in grand canonical ensemble [28].

Considering the solution (4.7) as the reference background, we get the Euclidean action difference between the charged black hole and the background

$$\mathcal{I} = \frac{\Omega_3 \beta}{16\pi G l^2}\left(2r_0^4 - 2r_{\rm max}^4 + k r_+^2 l^2 + \frac{q^2 l^2}{r_+^2} + r_+^4 - \frac{2q^2 l^2}{r_{\rm max}^2} + r_b^2 l^2 \delta_{k,-1}\right), \tag{4.8}$$

where Ω_3 is the volume of the three-dimensional closed surface $d\Omega_3^2$.

We first discuss the case without the IR cutoff. In this case, $r_0 = 0$, and then $r_{\rm max} = r_+$. The action becomes

$$\begin{aligned} \mathcal{I} &= -\frac{\Omega_3 \beta}{16\pi G l^2}\left(r_+^4 - k r_+^2 l^2 + \frac{q^2 l^2}{r_+^2} - r_b^2 l^2 \delta_{k,-1}\right), \\ &= -\frac{\Omega_3 \beta}{16\pi G l^2}\left(r_+^4 - r_+^2 l^2 (k - c^2\Phi^2) - r_b^2 l^2 \delta_{k,-1}\right). \end{aligned} \tag{4.9}$$

Obviously, when $k = 0$ or -1, the action is always negative. Therefore the dual field theory is in the deconfinement phase and the Hawking-Page phase transition does not occur here. On the other hand, when $k = 1$, the action changes its sign at

$$r_+^2 = r_c^2 = l^2(1 - c^2\Phi^2), \tag{4.10}$$

provided $\Phi^2 < 1/c^2$, which implies that the Hawking-Page phase transition happens at the critical temperature

$$T_{\rm HP} = \frac{3}{2\pi l}\sqrt{1 - c^2\Phi^2}. \tag{4.11}$$

When $\Phi^2 > 1/c^2$, the action is also always negative even if $k = 1$. In that case, the dual field theory is in the deconfinement phase and the confinement/deconfinement phase transition will not occur.

Next we consider the case with an IR cutoff r_0. And we will discuss separately the cases with $k = 1$, 0 and -1 below.

4.1 Case $k = 0$: Ricci flat black holes

The dual field theory to this gravitational background is a Yang-Mills theory at finite chemical potential and at finite temperature living on a flat four dimensional spacetime. For the gravitational computation in this case one has two subcases: $r_0 > r_+$ or $r_0 < r_+$.

(i) When $r_0 > r_+$, one has $r_{\rm max} = r_0$. In this case, the action reduces to

$$\mathcal{I} = \frac{\Omega_3 \beta}{16\pi G l^2}\left(r_+^4 + c^2\Phi^2 l^2 r_+^2(1 - 2r_+^2/r_0^2)\right). \tag{4.12}$$

The condition for the existence (4.4) of the horizon requires $r_+ > c\Phi l/\sqrt{2}$. We find that when $c\Phi l/\sqrt{2} < r_+ < r_0$, the action (4.12) is always positive. As a result, no phase transition happens in this case.

(ii) When $r_0 < r_+$, we have $r_{\rm max} = r_+$, and the action is

$$\mathcal{I} = \frac{\Omega_3 \beta}{16\pi G l^2}\left(2r_0^4 - r_+^4 - c^2\Phi^2 l^2 r_+^2\right). \tag{4.13}$$

The action changes its sign from positive to negative when the horizon radius crosses

$$r_c^2 = \frac{-c^2\Phi^2 l^2 + \sqrt{c^4\Phi^4 l^4 + 8r_0^4}}{2}. \tag{4.14}$$

Note from (4.4) that there exists a minimal horizon radius $r_{\rm min} = c\Phi l/\sqrt{2}$. we see from (4.14) that there is a gap for the IR cutoff. Namely the cutoff r_0 must satisfy

$$\frac{r_0^4}{l^4} > \frac{3}{8}c^4\Phi^4. \tag{4.15}$$

Clearly, this gap disappears when the charge is absent [26]. The Hawking-Page phase transition temperature is

$$T_c = \frac{1}{2\pi r_c l^2}\left(-2c^2\Phi^2 l^2 + \sqrt{c^4\Phi^4 l^4 + 8r_0^4}\right). \tag{4.16}$$

Clearly, the phase transition occurs due to the introduction of the IR cutoff.

4.2 Case $k = 1$: spherical black hole

(i) When $r_0 > r_+$, one has $r_{\rm max} = r_0$ and the reduced action is

$$\mathcal{I} = \frac{\Omega_3 \beta}{16\pi G l^2}\left(r_+^4 + r_+^2 l^2 + c^2\Phi^2 l^2 r_+^2(1 - 2r_+^2/r_0^2)\right). \tag{4.17}$$

Considering the condition (4.4) for the existence of the black hole horizon, once again, we find that the action is always positive. Therefore, no phase transition will occur in this case.

(ii) When $r_0 < r_+$, we have $r_{\rm max} = r_+$, and the Euclidean action is

$$\mathcal{I} = \frac{\Omega_3 \beta}{16\pi G l^2}\left(2r_0^4 - r_+^4 + r_+^2 l^2(1 - c^2\Phi^2)\right). \tag{4.18}$$

From the action, we can see that the Hawking-Page phase transition happens when the horizon radius crosses

$$r_c^2 = \frac{l^2(1-c^2\Phi^2) + \sqrt{(1-c^2\Phi^2)^2 l^4 + 8r_0^4}}{2}. \tag{4.19}$$

And the corresponding temperature at which the phase transition occurs is

$$T_c = \frac{1}{2\pi r_c}\left(2(1-c^2\Phi^2) + \sqrt{(1-c^2\Phi^2)l^4 + 8r_0^4}\right). \tag{4.20}$$

This temperature is higher than the one (4.11) for the case without the IR cutoff. Again, there is a gap for the IR cutoff because of the existence of the minimal horizon radius: the IR cutoff must be

$$\frac{r_0^4}{l^4} > \frac{3}{8}(1-c^2\Phi^2)^2. \tag{4.21}$$

4.3 Case $k=-1$: hyperbolic black hole

(i) When $r_0 > r_+$, we have $r_{\max} = r_0$, and the action reduces to

$$\mathcal{I} = \frac{\Omega_3\beta}{16\pi G l^2}\left(r_+^4 - r_+^2 l^2 + c^2\Phi^2 r_+^2 l^2(1 - 2r_+^2/r_0^2) + r_b^2 l^2\right). \tag{4.22}$$

According to the analysis in the previous section, we see that choosing the solution (3.4) with $r_b = 0$ as the reference background is not reasonable. Therefore in this subsection we focus on the case with $r_b = l/2$. Once again, within the range $r_{\min} = l\sqrt{(1+c^2\Phi^2)/2} < r_+ < r_0$, the action is found to be always positive and no phase transition will occur, dual QCD is in the confinement phase.

(ii) When $r_0 < r_+$, one has $r_{\max} = r_+$. The action reduces to

$$\mathcal{I} = \frac{\Omega_3\beta}{16\pi G l^2}\left(2r_0^4 - r_+^4 - r_+^2 l^2 - c^2\Phi^2 l^2 r_+^2 + r_b^2 l^2\right). \tag{4.23}$$

Clearly when $r_0 = 0$, no phase transition will occur. However, when $r_0 \neq 0$, the Hawking-Page phase transition happens when the horizon crosses r_c, satisfying

$$r_c^2 = \frac{(1+c^2\Phi^2)l^2}{2}\left(-1 + \sqrt{1 + \frac{8r_0^4 + 4r_b^2 l^2}{(1+c^2\Phi^2)l^4}}\right). \tag{4.24}$$

Since the critical radius must be larger than the minimal horizon radius $r_{\min}$, this leaves us with a gap for the IR cutoff

$$\frac{r_0^4}{l^4} > \frac{2+3c^2\Phi^2}{8}. \tag{4.25}$$

Compared to the case (3.13) without charge, the IR cutoff gap increases. Furthermore, the critical temperature of the phase transition is

$$T_c = \frac{1}{2\pi r_c}\left(-2(1+c^2\Phi^2)l^2 + \sqrt{1+\frac{8r_0^4+4r_b^2l^2}{(1+c^2\Phi^2)^2l^4}}\right). \tag{4.26}$$

5 Conclusions

In this work we have studied the thermodynamics and Hawking-Page phase transition of a hard wall AdS/QCD model in curved spaces by introducing an IR cutoff, generalizing the work in [26, 27], where the authors have discussed the phase transition between the low temperature AdS space and the high temperature AdS black holes in Poincare coordinates, which implies that the dual field theory resides on a Ricci flat space. In our case, the dual field theory lives in a curved space with positive or negative constant curvature, dual to black hole configurations having spherical or hyperbolic horizons.

In the case of the spherical AdS black holes, introducing the IR cutoff leads the Hawking-Page phase transition temperature to increase, compared to the case without the IR cutoff. For the case of the hyperbolic black hole, the Hawking-Page phase transition will not occur when one does not introduce an IR cutoff, while the transition happens once the IR cutoff is introduced. However, there is a gap for the IR cutoff in this case. Below that gap, the Hawking-Page phase transition still does not occur.

For the charged AdS black holes with any horizon topology, like the case without charge, the Hawking-Page phase transition becomes possible due to the introduction of an IR cutoff. A remarkable feature in this case is that for any horizon topology, a gap for the IR cutoff always exists due to the existence of a minimal black hole horizon.

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